Linear algebra recitation VI

1. Find the best least-squares solution to $3x = 10, 4x = 5$. What is the error you are minimizing with your solution? Check that the error vector $(10 - 3x, 5 - 4x)^T$ is perpendicular to $(3, 4)^T$.

2. (a) Solve $Ax = b$ by least squares and find the closest vector $p = Ax$ for
   \[
   A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
   \]
   Verify that the error $b - p$ is perpendicular to the columns of $A$.

   (b) Do the same with the same $A$ and new $b = (1, 3, 4)^T$. Find the error $b - p$ again. What is weird?

3. Solve $Ax = b$ with
   \[
   A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{n \times 1}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.
   \]
   This was a rather complicated way to compute something simple. What is the resulting $x$? Explain why we got it.

4. Find the (orthogonal) projection of $b$ onto the column space of $A$:
   \[
   A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.
   \]
   Split $b$ into $p + q$, with $p$ in the column space and $q$ perpendicular to that space. Which of the four subspaces of $A$ contains $q$ (column, row, left/right null)?

5. Find the projection matrix $P$ onto the space spanned by
   \[
   a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.
   \]

6. If $P$ is the projection matrix onto a $k$-dimensional subspace $S$ of the whole space $\mathbb{R}^n$, what is the column space of $P$ and what is its rank?

7. (a) If $P = P^T P$, show that $P$ is a projection matrix.

   (b) What subspace does the matrix $P = 0$ project onto?
8. Assume that we have data points \((t_1, b_1), (t_2, b_2), \ldots, (t_k, b_k)\) and we would like to find a combination of the functions \(\sin(t)\) and \(\cos(t)\) that fits my data the best. Describe this problem with our closest vector method.

9. Compute \(AT\) and its eigenvalues \(\sigma_1^2, \sigma_2^2\) and unit eigenvectors \(v_1, v_2\):

\[
A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}.
\]

10.

(a) Compute \(AA^T\) and its eigenvalues \(\sigma_1^2, \sigma_2^2\) and unit eigenvectors \(u_1, u_2\).

(b) Choose signs so that \(Av_i = \sigma_i u_i\) and verify the SVD:

\[
\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T
\]

(c) Which four vectors give orthonormal bases for the column space, row space, left and right nullspace of \(A\)?

11. Find the SVD from the eigenvectors \(v_1, v_2\) of \(AT\) and \(Av_i = \sigma_i u_i\) for the Fibonacci matrix

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]

12. Compute \(AT, AA^T\), and their eigenvalues and unit eigenvectors for

\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\]

Multiply \(U\Sigma V^T\) to recover \(A\).

13. Suppose \(u_1, \ldots, u_n\) and \(v_1, \ldots, v_n\) are orthonormal bases for \(\mathbb{R}^n\). Construct the matrix \(A\) that transforms each \(v_j\) into \(u_j\) to give \(Av_j = u_j\).

14. Suppose \(A\) is a 2 by 2 symmetric matrix with unit eigenvectors \(u_1\) and \(u_2\). If its eigenvalues are \(\lambda_1 = 3\) and \(\lambda_2 = -2\), what are \(U, \Sigma, \) and \(V^T\)?

15.

(a) If \(A\) changes to \(4A\), what is the change in the SVD?

(b) What is the SVD for \(A^T\) and for \(A^{-1}\)?

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*Please hand in the homeworks until Monday (19.11.) 5 pm to my folder (office building west 3rd floor to the left)*