Problem 1: The Legendre transform of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f^*(x) = \sup_{y \in \mathbb{R}} \{ xy - f(y) \}$$

Show that the double Legendre transform $f^{**}$ coincides with the convex and lower semicontinuous envelope of $f$ defined in the lecture.

Problem 2: For $f : \mathbb{R} \to \mathbb{R}_+$ and $\lambda > 0$, define

$$(T_\lambda f)(x) = \inf_{y \in \mathbb{R}} \{ f(y) + \lambda |y - x| \}$$

Show that $T_\lambda f$ is convex if $f$ is convex.

Problem 3: Show that functions in $W^{1,p}(\mathbb{R})$ with $p > 1$ are Hölder continuous.

Problem 4: Assume that $X$ is a vector space, and that $f_\infty = \Gamma\text{-lim}_{j \to \infty} f_j$. Show that if every $f_j$ is homogeneous of degree $\alpha > 0$, i.e., $f_j(tx) = t^\alpha f_j(x)$ for all $t > 0$ and $x \in X$, then $f_\infty$ has the same property.

Problem 5:

a) Show that if $f : \mathbb{R} \to \mathbb{R}_+$ is a convex function, its left and right derivatives exist at every point, and are actually equal except for possibly a countable set.

b) Show that if $f_j$ is a sequence of convex functions converging pointwise to $f$, then $f$ is convex. Moreover, the right and left derivatives of $f_j$ converge to the derivative of $f$ at all points where the latter exists.

Problem 6: Prove that the lower semicontinuous envelope with respect to $L^2((-1, 1))$-convergence of the functional

$$F(u) = \int_{-1}^{1} t^2 |u'(t)|^2 dt$$

is finite on $W^{1,2}((-1, 0) \cup (0, 1))$. 