Solution Sheet II

Introduction to Random Matrices, IST Austria
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1 Bound on largest eigenvalue

The goal of this exercise is to obtain a direct proof of the fact that the operator norm of random matrices with independent entries of size $1/\sqrt{N}$ is bounded. Because the argument is based on simple concentration estimates it is only applicable to the case of uniformly subgaussian matrix entries.

Definition. A centered random variable $X$ is called subgaussian if there exists $a > 0$ such that for all $t \in \mathbb{R}$ it holds that $E e^{tX} \leq e^{at^2/2}$. A family of random variables is called uniformly subgaussian if the same constant $a$ can be chosen for all random variables.

Problem 1. Prove that the following three assertions about a centered random variable $X$ are equivalent:

(i) $X$ is subgaussian,

(ii) there exists $b > 0$ such that for all $\lambda > 0$, $P(|X| \geq \lambda) \leq 2e^{-b\lambda^2}$,

(iii) there exists $c > 0$ such that $E e^{cX^2} \leq 2$.

Solution.

(i)$\Rightarrow$(ii) It follows from Markov’s inequality that

$$P(X \geq \lambda) \leq \frac{E e^{tX}}{e^{t\lambda}} \leq e^{at^2/2-t\lambda} \leq e^{-\lambda^2/2a}$$

and consequently $P(|X| \geq \lambda) \leq 2e^{-\lambda^2/2a}$.

(ii)$\Rightarrow$(iii) We find

$$E e^{cX^2} = \int_0^\infty P(e^{cX^2} \geq t) \, dt = 1 + \int_0^\infty 2c\lambda e^{c\lambda^2} P(|X| \geq \lambda) \, d\lambda \leq 1 + 4c \int_0^\infty \lambda e^{(c-b)\lambda^2} \, d\lambda$$

and the claim follows from choosing $c$ small enough.

(iii)$\Rightarrow$(i) Since $X$ is centered it follows that

$$E e^{tX} = 1 + \int_0^1 (1-r)(tX)^2 e^{rtX} \, dr \leq 1 + t^2 E X^2 e^{t|X|} \leq 1 + t^2 e^{t^2/2c} E X^2 e^{cX^2/2}$$

$$\leq 1 + 2t^2 e^{t^2/2c} E e^{cX^2} \leq 1 + 2t^2 e^{t^2/2c}$$

and the claim follows for some $a$ small enough. We note that in the second inequality we used the often useful weighted Cauchy-Schwarz inequality $|tX| = \left| \frac{t}{\sqrt{2}} (\sqrt{2}X) \right| \leq \frac{t^2}{2c} + \frac{cX^2}{2}$.

Let $H$ be a real $N \times N$ random matrix with independent entries $(h_{ij})_{i,j \in [N]}$ of zero mean such that the normalized random variables $\sqrt{N}h_{ij}$ are uniformly subgaussian. The goal of this exercise is to prove that there exists a constant $c > 0$ such that

$$P(\|H\| > C) \leq e^{-cNC^2}$$

for all $C$ large enough.
Problem 2. Let $x$ be a fixed vector of (Euclidean) length $\|x\| = 1$. Show that there exists a constant $c > 0$ such that
\[ P(\|Hx\| > C) \leq e^{-cNC^2} \] (2)
for all $C$ large enough.

Solution. We first observe that the random variables $\sqrt{N}(Hx)_i$ are uniformly subgaussian. Indeed,
\[ E e^{t\sqrt{N}(Hx)_i} = \prod_j E e^{t\sqrt{N}h_{ij}x_j} \leq \prod_j e^{a_t x_j^2/2} = e^{at^2/2}. \]
Thus it follows from Problem 1 that there exists a constant $c > 0$ such that $E e^{\sqrt{N}(Hx)_i^2} \leq 2$ for all $i$ and we can conclude from Markov’s inequality that
\[ P(\|Hx\| \geq C) \leq \frac{E e^{cN\|Hx\|^2}}{e^{cCN}} = \prod_i \frac{E e^{cN(Hx)_i^2}}{e^{cCN}} \leq \frac{2^N}{e^{cCN}} \]
and the claim follows for some slightly smaller $c$.

Problem 3. Let $P \subset S := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ be a maximal 1/2-separated set, i.e., a set for which for any $x \neq y \in P$ we have that $\|x - y\| \geq 1/2$ and for each $z \in S \setminus P$ there exist $x \in P$ such that $\|x - z\| < 1/2$. Prove that
\[ P(\|H \| > C) \leq P(\|Hx\| > C/2 \text{ for some } x \in P). \] (3)

Solution. By compactness we find $x \in S$ such that $\|H\| = \|Hx\|$. If $x \in P$, then (3) follows immediately. Otherwise we find $y \in P$ such that $\|x - y\| < 1/2$ from maximality. It follows that
\[ \|Hy\| = \|Hx + H(y - x)\| \geq \|H\| - \|H\|/2 = \|H\|/2 \]
and we conclude that $\|Hy\| > C/2$ for some $y \in P$ whenever $\|H\| > C$.

Problem 4. Combine the statements of problems 2 and 3 to prove 1.

Solution. Let $P$ be any maximal 1/2-separated set. Then the balls of radius 1/4 around each point $x \in P$ are disjoint and at the same time contained in the ball of radius 5/4 around the origin. It follows that $|P| \leq K^N$ for some constant $K$. Thus it follows that
\[ P(\|H\| \geq C) \leq \sum_{x \in P} P(\|Hx\| \geq C/2) \leq K^N e^{-cNC^2/4} \]
and we can choose $c$ appropriately such that 1 holds for $C$ large enough.

2 Interlacing eigenvalues

Let $A$ be a Hermitian $N \times N$ matrix and let $B$ be the $(N - 1) \times (N - 1)$ principal submatrix, i.e.,
\[ A = \begin{pmatrix} B & a \\ a^* & b \end{pmatrix} \]
for some vector $a \in \mathbb{C}^{N-1}$ and scalar $b \in \mathbb{R}$ and suppose that $A$ and $B$ have disjoint, simple spectra. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ and $\mu_1 < \cdots < \mu_{N-1}$ denote the ordered eigenvalues of $A$ and $B$. The goal of this exercise is to prove that the eigenvalues of $B$ interlace the eigenvalues of $A$, i.e., that
\[ \lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{N-1} < \mu_{N-1} < \lambda_N. \] (4)

We begin with an equivalent characterisation of interlacing.

Problem 5. Let $f, g$ be polynomials with distinct simple real roots and leading positive coefficients of degree $N$ and $N - 1$. Prove that the roots interlace if and only if for each $\lambda \in [0, 1]$ the polynomial $\lambda f + (1 - \lambda)g$ has only real roots.
as well as the Hermitian \( A \) to be \( \lambda \) is real. In the interval \( I_0 = (\nu_{2N-1}, \infty) \). In the interval \( I_1 = (\nu_{2N-2}, \nu_{2N-1}) \) for the polynomials \( f, g \) this gives different signs and continuing inductively we find that the polynomials \( f, g \) have equal signs in some interval \( I = (\nu_{2N-1-j}, \nu_{2N-j}) \) if and only if \( j \) is even.

Now suppose that \( h_\lambda := \lambda f + (1 - \lambda)g \) has real roots for all \( \lambda \) which can only lie in the intervals \( I_1, I_3, \ldots, I_{2N+1} \). The roots of \( h_\lambda \) are continuous functions of \( \lambda \) and thus cannot jump from one interval to another. By considering \( \lambda \) close to 0 and 1 it follows that for each odd interval the two interval endpoints have to be roots of \( f \) and \( g \), respectively. In particular it follows that the roots interlace.

Now suppose that the roots interlace. Then \( h_\lambda(\nu_{2N-1}) > 0, h_\lambda(\nu_{2N-3}) < 0, h_\lambda(\nu_{2N-5}) > 0 \) and so on and we find \( N \) real roots from the intermediate value theorem.

\[
\text{Problem 5} \text{ was wrong as stated. It turns out that the polynomials } f(x) = x^2 - 1 \text{ and } g(x) = x - 2 \text{ provide a counter example. Can you spot the mistake in the above argument? The correct version follows now.}
\]

**Problem 5.** Let \( f, g \) be polynomials with distinct simple real roots and leading positive coefficients of degree \( N \) and \( N - 1 \). Prove that the roots interlace if and only if for each \( \mu, \lambda \in \mathbb{R} \) the polynomial \( h = \lambda f + \mu g \) has only real roots.

**Solution.** Denote the ordered roots of \( f \) by \( \alpha_1 < \cdots < \alpha_N \). First assume that the roots are interlacing and fix \( \mu, \lambda \in \mathbb{R} \). Then \( \text{sgn } h_\lambda(\alpha_k) = \text{sgn } \mu g(\alpha_k) \) is alternating since the same is true for \( g \). Thus \( h \) has at least \( N - 1 \) real roots by the intermediate value theorem and as complex roots come in pairs of complex conjugates and \( \deg h = N \) it follows that all \( N \) roots of \( h \) are real.

Conversely, assume that \( h \) has only real roots for all \( \lambda, \mu \). Then the rational function \( \psi(x) = \lambda / \mu + g(x) / f(x) \) has a derivative \( \psi' \) of constant sign. Indeed, if not then we could find a root of \( \psi' \) and by choosing \( \lambda / \mu \) appropriately we could create a pair of complex roots of \( h = \mu \psi f \), in contradiction with the assumption. Thus the sign of \( g'f - f'g = f^2 \psi' \) is also constant and thus \( f'(\alpha_i)g(\alpha_i) \) has the same sign for all \( i \). As \( f'(\alpha_i) \) alternates in sign so does \( g(\alpha_i) \) and we can find a root of \( g \) between any two consecutive \( \alpha_i \).

It turns out that Problem 5 provides a useful characterisation of interlacing for proving (4).

**Problem 6.** Find a Hermitian matrix \( C = C(\lambda) \) such that

\[
\det(x - C) = \det(x - A) + \frac{\mu}{\lambda} \det(x - B)
\]

and conclude from Problem 5 that the eigenvalues of \( B \) interlace those of \( A \), i.e., (4).

**Solution.** By linearity of the determinant

\[
\det \begin{pmatrix} x - B & -a \\ -a^* & x - b + \frac{\mu}{\lambda} \end{pmatrix} = \det \begin{pmatrix} x - B & -a \\ -a^* & x - b \end{pmatrix} + \det \begin{pmatrix} x - B & -a \\ 0 & \frac{\mu}{\lambda} \end{pmatrix} = \det(x - A) + \frac{\mu}{\lambda} \det(x - B),
\]

where the polynomial on the lhs. is real-rooted as the characteristic polynomial of a Hermitian matrix. By multiplying both sides by \( \lambda \) it follows from Problem 5 that the roots of \( \det(x - B) \) interlace those of \( \det(x - A) \), proving the claim.

\section*{3 Resolvent identities}

The goal of this exercise is to prove two identities relating resolvent elements to those of the resolvent of certain minors. Let \( H \) be a Hermitian \( N \times N \) matrix. For \( i \in [N] \) let \( H^{(i)} \) denote the matrix with the \( i \)-th row and column set to zero, i.e., \( H_{ij}^{(i)} = H_{ik} 1_{i \neq k} 1_{i \neq j} \). The entries of the resolvent \( G^{(i)}(z) = (H^{(i)} - z)^{-1} \) of such a minor satisfy the first resolvent decoupling identity for \( i, j \neq k \)

\[
G_{ij} = G_{ij}^{(k)} + \frac{G_{ik}G_{kj}}{G_{kk}}
\]  

(5a)

as well as the second resolvent decoupling identity for \( i \neq j \)

\[
G_{ij} = -G_{ii} \sum_{k \neq i} h_{ik} G_{kj}^{(i)} = -G_{jj} \sum_{k \neq j} G_{ik}^{(j)} h_{kj},
\]  

(5b)

Before proving (5a)–(5b) we recall the standard resolvent expansion formula.
Problem 7. Show that for matrices $A, B$ it holds that
\[
(A - B)^{-1} = A^{-1} + A^{-1}B(A - B)^{-1} = A^{-1} + (A - B)^{-1}BA^{-1},
\] (6)
provided that all inverses exist.

Solution. We only prove the first equality as the proof for the second is identical. Multiplying both sides with $(A - B)$ from the right gives
\[
1 = 1 - A^{-1}B + A^{-1}B = 1
\]
and the claimed equality follows. \(\square\)

Problem 8. Use (6) to first prove (5b) and then use (6) and (5b) to prove (5a).

Solution. We first prove (5b) and again only prove the first equality as the proof of the second one is identical. It follows from (6) that
\[
G = (H - z)^{-1} = (H^{(i)} - z - [H^{(i)} - H])^{-1} = G^{(i)} + G(H^{(i)} - H)G^{(i)}
\]
and consequently
\[
G_{ij} = G_{ij}^{(i)} - \sum_k G_{ik}h_{kj}G_{kj}^{(i)} - \sum_k G_{ij}^{(i)}h_{ki}G_{ik}^{(i)} = -G_{ii} \sum_{k \neq i} h_{ik}G_{kij}^{(i)},
\]
where the second equality followed from $G_{ij}^{(i)} = 0$.

We now proceed with the proof of (5a). It follows from (6) that
\[
G = (H - z)^{-1} = (H^{(k)} - z - [H^{(k)} - H])^{-1} = G^{(k)} + G(H^{(k)} - H)G^{(k)}
\]
and therefore it follows from (5b) that
\[
G_{ij} = G_{ij}^{(k)} - \sum_{l \neq j} G_{ik}h_{kl}G_{lj}^{(k)} = G_{ij}^{(k)} + \frac{G_{ik}G_{kj}}{G_{kk}},
\]
completing the proof. \(\square\)