1 Bound on largest eigenvalue

The goal of this exercise is to obtain a direct proof of the fact that the operator norm of random matrices with independent entries of size \(1/\sqrt{N}\) is bounded. Because the argument is based on simple concentration estimates it is only applicable to the case of uniformly sub-gaussian matrix entries.

**Definition.** A centered random variable \(X\) is called subgaussian if there exists \(a > 0\) such that for all \(t \in \mathbb{R}\) it holds that 
\[
E e^{tX} \leq e^{at^2/2}.
\]
A family of random variables is called uniformly subgaussian if the same constant \(a\) can be chosen for all random variables.

**Problem 1.** Prove that the following three assertions about a centered random variable \(X\) are equivalent:

(i) \(X\) is subgaussian,

(ii) there exists \(b > 0\) such that for all \(\lambda > 0\), 
\[
P(|X| \geq \lambda) \leq 2e^{-b\lambda^2},
\]

(iii) there exists \(c > 0\) such that 
\[
E e^{cX^2} \leq 2.
\]

Let \(H\) be a real \(N \times N\) random matrix with independent entries \((h_{ij})_{i,j \in [N]}\) of zero mean such that the normalized random variables \(\sqrt{N}h_{ij}\) are uniformly subgaussian. The goal of this exercise is to prove that there exists a constant \(c > 0\) such that
\[
P(\|H\| > C) \leq e^{-cNC^2}
\] for all \(C\) large enough.

**Problem 2.** Let \(x\) be a fixed vector of (Euclidean) length \(\|x\| = 1\). Show that there exists a constant \(c > 0\) such that 
\[
P(\|Hx\| > C) \leq e^{-cNC^2}
\] for all \(C\) large enough.

**Problem 3.** Let \(P \subset S := \{x \in \mathbb{R}^N | \|x\| = 1\}\) be a maximal \(1/2\)-separated set, i.e., a set for which for any \(x \neq y \in P\) we have that \(\|x - y\| \geq 1/2\) and for each \(z \in S \setminus P\) there exist \(x \in P\) such that \(\|x - z\| < 1/2\). Prove that
\[
P(\|H\| > C) \leq P(\|Hx\| > C/2 \text{ for some } x \in P).
\]

**Problem 4.** Combine the statements of problems 2 and 3 to prove 1.

2 Interlacing eigenvalues

Let \(A\) be a Hermitian \(N \times N\) matrix and let \(B\) be the \((N-1) \times (N-1)\) principal submatrix, i.e.,
\[
A = \begin{pmatrix} B & a \\ a^* & b \end{pmatrix}
\]
for some vector \(a \in \mathbb{C}^{N-1}\) and scalar \(b \in \mathbb{R}\) and suppose that \(A\) and \(B\) have disjoint, simple spectra. Let \(\lambda_1 < \lambda_2 < \cdots < \lambda_N\) and \(\mu_1 < \cdots < \mu_{N-1}\) denote the ordered eigenvalues of \(A\) and \(B\). The goal of this exercise is to prove that the eigenvalues of \(B\) interlace the eigenvalues of \(A\), i.e., that
\[
\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{N-1} < \mu_{N-1} < \lambda_N.
\]

We begin with an equivalent characterisation of interlacing.
Problem 5. Let \( f, g \) be polynomials with distinct simple real roots and leading positive coefficients of degree \( N \) and \( N - 1 \). Prove that the roots interlace if and only if for each \( \lambda \in [0, 1] \) the polynomial \( \lambda f + (1 - \lambda)g \) has only real roots.

It turns out that Problem 5 provides a useful characterisation of interlacing for proving (4).

Problem 6. Find a Hermitian matrix \( C = C(\lambda) \) such that
\[
\det(x - C) = \det(x - A) + \frac{1 - \lambda}{\lambda} \det(x - B)
\]
and conclude from Problem 5 that the eigenvalues of \( B \) interlace those of \( A \), i.e., (4).

3 Resolvent identities

The goal of this exercise is to prove two identities relating resolvent elements to those of the resolvent of certain minors. Let \( H \) be a Hermitian \( N \times N \) matrix. For \( i \in [N] \) let \( H^{(i)} \) denote the matrix with the \( i \)-th row and column set to zero, i.e., \( H^{(i)}_{kl} = H_{kl} \cdot (i \neq k \land i \neq l). \) The entries of the resolvent \( G^{(i)}(z) = (H^{(i)} - z)^{-1} \) of such a minor satisfy the first resolvent decoupling identity for \( i, j \neq k \)
\[
G_{ij} = G_{ij}^{(k)} + \frac{G_{ik}G_{kj}}{G_{kk}}
\] (5a)
as well as the second resolvent decoupling identity for \( i \neq j \)
\[
G_{ij} = -G_{ii} \sum_{k \neq i} h_{ik} G_{kj}^{(i)} = -G_{jj} \sum_{k \neq j} G_{ik}^{(j)} h_{kj}.
\] (5b)

Before proving (5a)-(5b) we recall the standard resolvent expansion formula.

Problem 7. Show that for matrices \( A, B \) it holds that
\[
(A - B)^{-1} = A^{-1} + A^{-1} B (A - B)^{-1} = A^{-1} + (A - B)^{-1} B A^{-1},
\] (6)
provided that all inverses exist.

Problem 8. Use (6) to first prove (5b) and then use (6) and (5b) to prove (5a).