1. Consider the following subsets of the vector space \( \mathbb{R}^2 \):

(a) \( P = \{(t, t^2) : t \in \mathbb{R}\} \).

(b) \( C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 4\} \).

(c) \( L_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 - x_2 = 0\} \).

(d) \( L_2 = \{(0, t) : t \in \mathbb{R}\} \).

(e) \( L_3 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - 2x_2 = 3\} \).

Draw a picture of each of these sets. Which of these sets are linear subspaces of \( \mathbb{R}^2 \)? (Justify your answer in each case)

2. Consider the following four types of linear subspaces \( U \) of \( \mathbb{R}^3 \):

0. \( U = \{0\} \)

1. \( U = L(v) \) for some nonzero vector \( v \in \mathbb{R}^3 \).

2. \( U = L(v_1, v_2) \) for two vectors \( v_1, v_2 \in \mathbb{R}^3 \) such that \( \{v_1, v_2\} \) is linearly independent.

3. \( U = \mathbb{R}^3 \).

Describe these types of subspaces in intuitive geometric terms. Moreover, try to reason that every linear subspace of \( \mathbb{R}^3 \) is of one of these four types.

3. Which of the following statements is correct? The vector space \( V = \{0\} \) consisting only of the zero vector

(a) has the basis \( \{0\} \)

(b) has the basis \( \emptyset \)

(c) has no basis.

4. Which of the following statements is correct? If \( \{v_1, v_2, v_3\} \) is a linearly independent set of vectors in a vector space \( V \) then

(a) \( \{v_1, v_2\} \) is always linearly dependent

(b) \( \{v_1, v_2\} \) may be linearly dependent or linearly independent, depending on the choice of \( \{v_1, v_2, v_3\} \)

(c) \( \{v_1, v_2\} \) is always linearly independent.

5. (a) Consider the following five vectors in \( \mathbb{R}^2 \): 

\[
\begin{align*}
  v_1 &= \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \\
  v_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
  v_3 &= \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \\
  v_4 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \\
  v_5 &= \begin{pmatrix} -1 \\ -1 \end{pmatrix}
\end{align*}
\]

Which subsets of \( \{v_1, v_2, v_3, v_4, v_5\} \) are linearly independent? Which subsets are bases of \( \mathbb{R}^2 \)? (Draw a picture to help your intuition, but justify your answers formally.)
(b) Consider the following five vectors in $\mathbb{R}^3$:

\[ w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \]

Which subsets of \{w_1, w_2, w_3, w_4, v_5\} are linearly independent? Which subsets are bases of $\mathbb{R}^3$?

6. Let $V$ be a vector space and $S$ any subset of $V$. Prove that the linear span $L(S)$ is a linear subspace of $V$.

7. What is the dimension of the vector space of 3x3 magic squares (row and column sums are equal)? Try to find a basis of it.

8. * Let $V$ be a vector space.
   (a) Let $U$ be a linear subspace of $V$ and suppose that $V$ is finite-dimensional. Show that $U$ is also finite-dimensional and that $\dim(U) \leq \dim(V)$. Furthermore, show that $\dim(U) < \dim(V)$ if and only if $U \neq V$.
   (b) Let $U$ and $W$ be linear subspaces of $V$. Show that
   \[ U + W := \{ u + w : u \in U, w \in W \} \]
   is a linear subspace of $V$.
   (c) Let $U$ and $W$ be linear subspaces of $V$. Show that the intersection $U \cap W$ is a linear subspace of $V$.
   (d) Suppose that $V$ is a vector space, that $U$ and $W$ are linear subspaces of $V$, and that furthermore, $V$ is finite-dimensional. Show that
   \[ \dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W) \]

9. * Consider the vector space of all polynomials with real coefficients in one variable $x$,
   \[ f(x) = a_0 + a_1x + \ldots + a_nx^n \]
   \((n \in \mathbb{N}, a_0, a_1, \ldots, a_n \in \mathbb{R})\). This vector space is often denoted $\mathbb{R}[x]$.
   A particular type of polynomial are *monomials*, which are polynomials of the form
   \[ f(x) = x^k \]
   for some $k \in \mathbb{N}_0$ (i.e., polynomials with just one nonzero term, which has coefficient 1).
   Another important type of polynomials are the so-called *falling factorials*
   \[ x^k := x(x-1)(x-2)\cdots(x-k+1), \]
   $k \in \mathbb{N}_0$, with the convention that $x^0 = 1$.
   Show that the (infinite) sets
   \[ B_1 = \{1, x, x^2, x^3, \ldots\} \text{ and } B_2 = \{1, x^1, x^2, x^3, \ldots\} \]
   are bases of $\mathbb{R}[x]$.
   Please hand in the homeworks until Monday (23.10.) 5pm to my folder (office building west 3rd floor to the left).

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1We recall that, by definition, $U \cap W$ consists of all vectors $v \in V$ that lie in both $U$ and $W$. 